Fluctuations in the Moon's Mean Motion.
By Simon Newcomb. (Plate 11.)

With the aid of my assistant Dr. Frank E. Ross, I have brought to a completion a study of the Moon's mean motion based on observations having an extreme range in time of about 2600 years. The data of observation are as follows:—

1. The eclipses of the Moon found in Ptolemy's Almagest, observed between B.C. 720 and A.D. 134.

2. Observations of eclipses by the Arabian astronomers, extending from 829 to 1004.

3. Observations of eclipses of the Sun and of occultations of stars by the Moon made by Gassendi, Hevelius, and others, from 1620 to 1680.

4. Observations of occultations of stars from 1670 until the present time.

The observations previous to 1750 were all worked up in my *Researches on the Motion of the Moon*, published in 1878. I have, however, subjected the results to a careful revision, and grouped them in a slightly different way from the former one. From and after 1680 the observations are of a fair degree of precision, but there are frequent gaps during the last half of the 18th century. The observations are fairly continuous since 1820.

Taken in connection with the recent exhaustive researches of Brown, which seem to be complete in determining with precision the action of every known mass of matter upon the Moon, the present study seems to prove beyond serious doubt the actuality of the large unexplained fluctuations in the Moon's mean motion to which I have called attention at various times during the past forty years. In the *Monthly Notices* for March 1903 is found a comprehensive review of the whole problem so far as it had then been developed, so that I need not enter into details at present. Indeed, the general conclusions reached in the work of 1878 have only been slightly modified in the present one.

The feature of most interest is the great fluctuation with a period of between 250 and 300 years. I call this a fluctuation rather than an inequality because, in the absence of any physical cause for its continuance, there is no reason to suppose that it will continue in the future in accordance with the law followed in the past. In the former paper I found it convenient to represent it as of the same period with the great Hansenian inequality due to the action of Venus. Singularly enough, the present research shows that the period which best represents it is still the same as that of the Hansenian inequality. In the former work I showed that the observed fluctuation could be represented by a mere change of sign of the constant term in the argument of this inequality.

Putting

$$A = 18V - 16E - g$$

Hansen's value of this term is

$$15''\cdot34 \sin (A + 30^\circ\cdot2)$$

The empirical term found by the writer in 1877 to best represent the observations was

$$-15''\cdot5 \cos A$$

The sum of these gives a term having practically the same coefficient with an argument differing little from $A - 30^\circ$. A natural suspicion would have been that an error of sign had crept into the theory. But this is disproved by the fact that the constant in question is a simple function of the longitude of the node of Venus, the relation of which to the inequality, in theory at least, admits of no doubt.

The unaccounted-for fluctuation as now found is best represented by the term

$$12''\cdot95 \sin [1^\circ\cdot31(t - 1800) + 100^\circ\cdot6]$$

The argument of the Hansenian term is

$$A = 1^\circ\cdot32\ (t - 1800) + 183^\circ\cdot9$$

Practically, the annual motion $1^\circ\cdot32$ will represent observations as well as $1^\circ\cdot31$. We may therefore write the empirical term thus—

$$\Delta\lambda = 12''\cdot95 \sin (A - 83^\circ\cdot3)$$

The former empirical term was, as above,

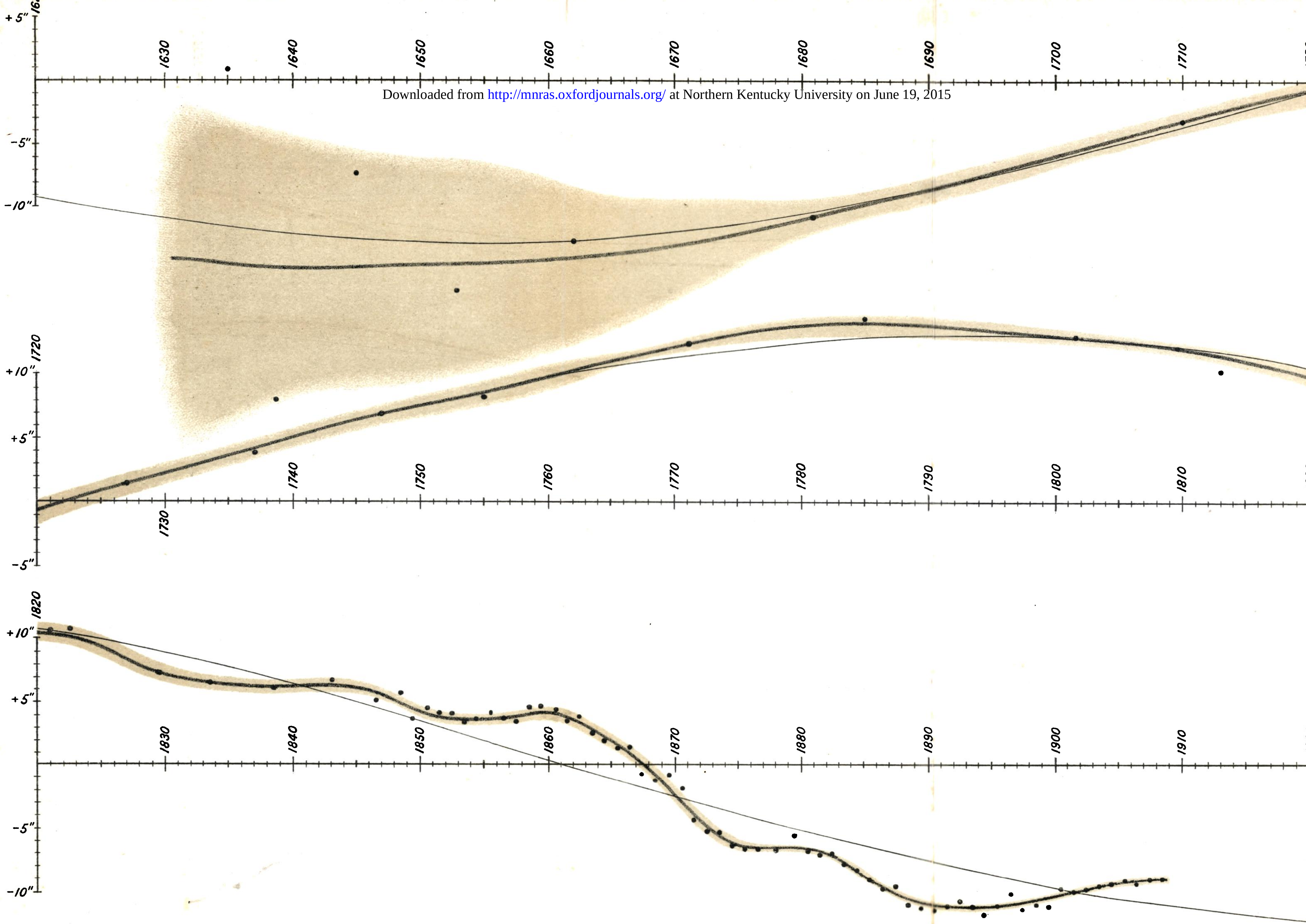
$$15''\cdot5 \sin (A - 90^\circ)$$

The following table shows the residual differences between the result of observations of the Moon since 1620 and pure gravitational theory. In deriving the elements of mean motion, it was necessary to divide the residual excess into two parts, one the great fluctuation just described, the other the smaller fluctuations which were superimposed upon it. In the table, the second column gives the minor fluctuations, which are in fact the residuals of the conditional equations. The third column shows the main fluctuation as computed from the expression given above. The sum of the two found in the fourth column, is the total excess of the Moon's observed longitude over the result of gravitational theory. It is, however, to be remarked that in this theory is included the excess of the observed over the theoretical secular acceleration.

The unit of weight, as the latter is given in the last column, corresponds to a probable error of about $\pm 0''\cdot9$, and a mean error of about $\pm 1''\cdot3$. Being in many cases partly a matter of judgment, round numbers are preferred where it is doubtful. The limiting value assigned is 60, it being judged that the actual probable error can never be below $\pm 0''\cdot12$.

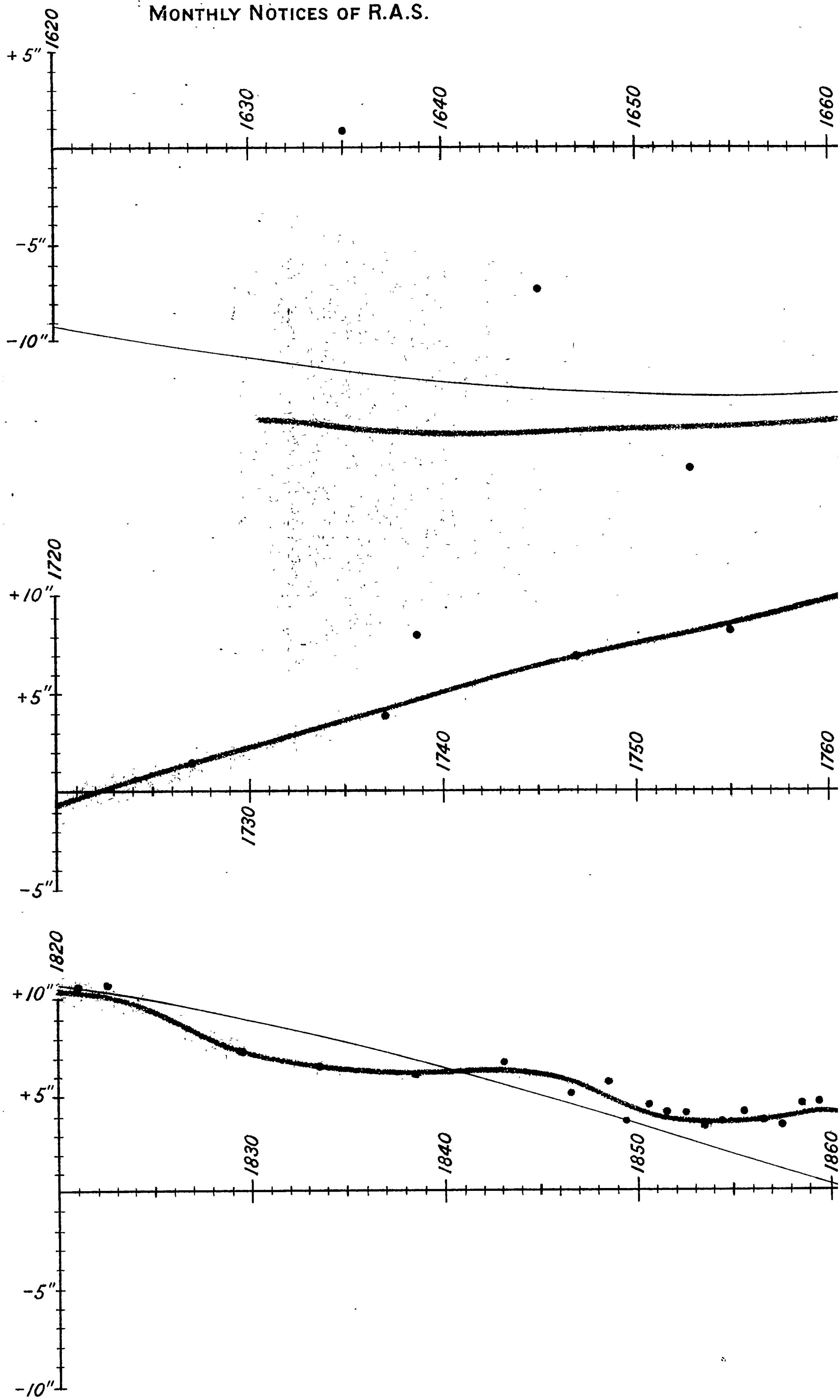
Mean Date.	Minor Res.	Great Fluctuation.	Total Fluctuation.	Weight.
	″	″	″	
1621	+39	− 9·6	+29	·005
1635	+13	−11·7	+ 1	·02
1639	−13	−12·1	−25	·04
1645	+ 5	−12·6	− 8	·03
1653	− 4	−12·9	−17	·03
1662	0	−12·7	−13	·06
1681	− 0·4	−10·5	−10·9	2
1710	+ 0·5	− 3·7	− 3·2	6
1727	+ 0·1	+ 1·2	+ 1·3	3
1737	− 0·2	+ 4·1	+ 3·9	6
1747	0·0	+ 6·8	+ 6·8	5
1755	− 0·3	+ 8·7	+ 8·4	2
1771	+ 1·4	+11·2	+12·6	9
1784·7	+ 1·6	+12·7	+14·3	5
1792	+ 0·3	+12·9	+13·2	10
1801·5	− 0·6	+12·7	+12·1	12
1809·5	− 0·1	+11·9	+11·8	16
1813	− 1·2	+11·5	+10·3	16
1821	+ 0·1	+10·3	+10·4	14
1822·5	+ 0·6	+10·0	+10·6	10
1829·5	− 1·6	+ 8·6	+ 7·0	20
1833·5	− 1·6	+ 7·7	+ 6·1	10
1838·5	− 0·6	+ 6·4	+ 5·8	30
1843	+ 1·1	+ 5·2	+ 6·3	20
1846·5	+ 1·1	+ 4·2	+ 5·3	10
1848·5	+ 1·9	+ 3·7	+ 5·6	8
1849·5	+ 0·1	+ 3·4	+ 3·5	15
1850·5	+ 1·0	+ 3·2	+ 4·2	18
1851·5	+ 0·8	+ 2·9	+ 3·7	12
1852·5	+ 0·9	+ 2·6	+ 3·5	8
1853·5	+ 0·7	+ 2·3	+ 3·0	7
1854·5	+ 1·4	+ 2·0	+ 3·4	14
1855·5	+ 2·1	+ 1·7	+ 3·8	6
1856·5	+ 1·9	+ 1·4	+ 3·3	6
1857·5	+ 2·1	+ 1·1	+ 3·2	9
1858·5	+ 3·5	+ 0·8	+ 4·3	8
1859·5	+ 3·9	+ 0·5	+ 4·4	6
1860·5	+ 3·9	+ 0·2	+ 4·1	12
1861·5	+ 3·3	− 0·1	+ 3·2	5
1862·5	+ 3·9	− 0·4	+ 3·5	7
1863·5	+ 3·0	− 0·7	+ 2·3	6
1864·5	+ 2·9	− 1·0	+ 1·9	10
1865·5	+ 2·6	− 1·3	+ 1·3	5

Mean Date.	Minor Res.	Great Fluctuation.	Total Fluctuation.	Weight.
	″	″	″	
1866·5	+2·8	− 1·6	+ 1·2	6
1867·5	+1·1	− 1·9	− 0·8	5
1868·5	+1·0	− 2·2	− 1·2	10
1869·5	+1·6	− 2·5	− 0·9	9
1870·5	+0·7	− 2·8	− 2·1	6
1871·5	−1·3	− 3·1	− 4·4	10
1872·5	−1·8	− 3·3	− 5·1	16
1873·5	−1·8	− 3·6	− 5·4	12
1874·5	−2·2	− 3·9	− 6·1	8
1875·5	−2·3	− 4·2	− 6·5	8
1876·5	−2·1	− 4·4	− 6·5	30
1878·0	−1·8	− 4·8	− 6·6	18
1879·5	−0·5	− 5·2	− 5·7	14
1880·5	−1·4	− 5·5	− 6·9	20
1881·5	−1·6	− 5·7	− 7·3	12
1882·5	−1·4	− 6·0	− 7·4	8
1883·5	−2·2	− 6·2	− 8·4	7
1884·5	−2·1	− 6·5	− 8·6	30
1885·5	−2·5	− 6·7	− 9·2	50
1886·5	−2·8	− 7·0	− 9·8	18
1887·5	−2·6	− 7·2	− 9·8	20
1888·5	−3·5	− 7·5	−11·0	8
1889·5	−3·5	− 7·7	−11·2	7
1890·5	−3·4	− 8·0	−11·4	10
1891·5	−3·1	− 8·2	−11·3	15
1892·5	−2·6	− 8·4	−11·0	17
1893·5	−2·7	− 8·6	−11·3	15
1894·5	−3·0	− 8·8	−11·8	30
1895·5	−2·2	− 9·0	−11·2	60
1896·5	−1·2	− 9·2	−10·4	60
1897·5	−2·0	− 9·5	−11·5	20
1898·5	−1·5	− 9·7	−11·2	28
1899·5	−1·4	− 9·9	−11·3	12
1900·5	0·0	−10·1	−10·1	15
1901·5	−0·1	−10·3	−10·4	16
1902·5	+0·3	−10·5	−10·2	18
1903·5	+0·6	−10·6	−10·0	12
1904·5	+1·1	−10·8	− 9·7	20
1905·5	+1·5	−11·0	− 9·5	20
1906·6	+1·3	−11·1	− 9·8	16
1907·5	+2·0	−11·3	− 9·3	15
1908·2	+2·1	−11·4	− 9·3	9



West, Newman lith.

Unexplained Fluctuations in the Moon's mean Longitude: 1630–1908.



UNEXPLAINED FLUCTUATIO

The observed secular acceleration is now found to be less by $0''.37$ than that which I derived in 1876. As for the theoretical value, I have added $0''.27$ to the value found by Brown and myself, on account of the effect due to the combination of the Earth's oblateness with the secular diminution of the obliquity of the ecliptic. This carries the theoretical acceleration up to $6''.08$. The value now found from all observations is

Sec. acc. from mean equinox	$9''.07$
Sidereal value	$7''.96$
Tidal excess	$1''.88$

The accompanying plate (Plate 11) gives a graphical representation in three sections of the residual deviations from pure theory, the motion derived from gravitational theory being represented by the straight medial lines. In order to show clearly the two parts into which the total fluctuation is divided, the term of great fluctuation is represented by a fine, sharp curve. The curve of actual longtitude is bounded on each side by a shaded area showing the mean error at each point, which is nearly $\frac{3}{2}$ of the probable error. In this way not only the fluctuations as shown by observation are exhibited, but also the error to which the curve may be subject, the probability being $\frac{2}{3}$ that at any point the true curve lies inside the shaded area, and $\frac{1}{3}$ that it lies without it.

We see by the curve as well as by the numbers, that before 1750 the observations are not sufficiently continuous, numerous, and accurate to show any fluctuation with certainty. The first minor fluctuation fairly well shown began about 1760. During the years 1765–1784 the Moon ran ahead by about $1''$. Then the excess of motion ceased, and became temporarily reversed. Since 1820 the motion has been marked by rapid fluctuations, which can be so well traced on the plate that no description is necessary.

Since what is actually observed is neither the acceleration nor the speed of motion, but changes of the longitude itself, of which these quantities are respectively the second and the first derivatives as to the time, it is not possible from the observations to make any approach to an accurate estimate of the accelerating or retarding forces. The most that we can say is that these varying forces are sufficient to bring about a change of annual motion amounting to between $0''.5$ and $1''.0$ by acting during a period of perhaps from 4 to 6 years.

It must be remarked that the separation of the entire deviation into two parts, one the great fluctuation of long period, the other minor fluctuations superimposed upon the great one, is made merely for convenience in representing past observations, and does not imply a corresponding duplicity in the causes at play. We know that these causes have acted in a certain way in the past 250 years, but we cannot infer with confidence that they will act in the same way in the future. In other words, we cannot confidently predict

a repetition of the great fluctuation through the next 250 or 300 years. Were there no minor fluctuations whatever, the belief that the great fluctuation was permanent might have some foundation, our conclusion then being that some natural cause was in action having the period in question. But, in the actual state of things, we have no reason to believe that the close correspondence between the observed motion and the great harmonic fluctuation is more than accidental. The fact is that the variations of accelerating force necessary to produce the minor fluctuations are much greater than the forces necessary to produce the great one, the measure of this force being, not the actual deviation, but the degree of curvature at each point of the line representing the path.]

The minor deviations during the past 100 years may be empirically represented by a trigonometric series, the principal term of which would have a period of 60 years, more or less, and an amplitude of perhaps 3″. But we have no reason to believe that, how accurate soever the representation may be by such a series, it will represent the future course of the Moon.

It would be of interest to compare the present curve from occultations with the deviations derived by Mr. Cowell from the Greenwich meridian observations, which he has represented by a trigonometric series. But I deem it desirable that this interesting comparison, and the conclusions to be drawn from it, should be the work of someone else.

I regard these fluctuations as the most enigmatical phenomenon presented by the celestial motions, being so difficult to account for by the action of any known causes, that we cannot but suspect them to arise from some action in nature hitherto unknown. A brief résumé of possible causes, and the difficulties in accepting them, may be attempted.

Taking it for granted that the gravitation of all known bodies has been allowed for in the comparison, and that no unknown bodies exist, the first explanation to occur to us is that the inequalities are only apparent, being perhaps due to fluctuations in the Earth's speed of rotation, and therefore in our measure of time.

I suggested this explanation in my earlier papers on the subject. It is open to the objection that it seems scarcely possible to account for changes so large as would be required through the action of known causes. But the explanation admits of an independent test from observation. If the fault is with our measure of time, it can be detected by the transits of Mercury and by the eclipses of the first satellite of Jupiter. As to the first, the discussions of the transits which I have already published, extending up to 1894, seem to preclude the possibility of any such changes in the measure of time as would account for the phenomena. The recent transit of 1907 November 14, which I have worked up in a preliminary way, seems conclusive on this point, since it would show our measure of time to be about 7 seconds slow, whereas to account for the observations of the Moon it should be more than this

amount fast. I am now engaged in the working up of observations of the first satellite of Jupiter, which may throw additional light on the subject.

A tidal friction varying with ocean currents, ice, and meteorological conditions may suggest itself. To this there is a double objection. Accepting as complete the received theory, the only effect of tidal friction would be through a tidal couple acting between the Earth and Moon. Granting the completeness of the theory, this couple could only result in the doing of work upon the rotating Earth by the Moon, and never in the Earth doing work upon the Moon, because in this case the friction would have to be a negative quantity. But apart from this, the effect of any tidal couple would be to produce wider fluctuations in the Earth's rotation, and therefore in our measure of time, than those which would by themselves account for the fluctuations in the Moon's apparent motion. At the same time it is worthy of remark that the current theory of tidal friction, and the corresponding couple, is incomplete, in that it takes no account of the tide-producing action of the Sun, which it seems quite natural to consider as incapable of modifying the lunar couple. But this should be investigated, not assumed.

The preceding suggestions seem to me to include every known cause of action. If we pass to unknown causes and inquire what is the simplest sort of action that would explain all the phenomena, the answer would be—a fluctuation in the attraction between the Earth and Moon. Accepting the law of the conservation of energy, such a fluctuation would involve an expenditure or absorption of energy somewhere in the solar system, which it seems difficult to admit. Precisely what changes of gravitation would be required I have not yet computed; but it seems quite likely that they would be below any that could be determined by experimental methods on the Earth. It would be natural to associate them with the Sun's varying magnetic activity and the varying magnetism of the Earth; but I cannot find that we have any data on this subject which would enable us to base any law upon varying magnetic action. At present I see nothing more to do than to invite the attention of investigators to this most curious subject.

One general result of the present state of things is that we cannot draw any precise conclusions from a discussion of the Moon's motion in longitude, how refined soever we make it. For example, it is impossible to derive from observation the accurate coefficient of the 18·6-year nodal inequality in longitude, owing to the varying fluctuation.

It is also not possible to predict the future motion of the Moon with precision. If we require our ephemerides of the Moon's longitude to be as exact as possible, we must correct the tabular mean longitude from time to time by observations.

Washington:
1908 *December* 11.

Development of the Disturbing Function in Planetary Theory, in terms of the mean anomalies and constant elliptic elements. By P. H. Cowell, M.A., F.R.S.

1. The following notation will be employed :—

a mean distance
f true anomaly
r radius vector
g mean anomaly
$\epsilon = \tan \frac{1}{2} \psi$ where $\sin \psi$ is the eccentricity.

Plain letters refer to the disturbed planet, accented letters to the disturbing planet.

ω, ω' the distance from the intersection of the orbits to the perihelia.

σ = sine of half the inclination of the two orbit planes.

2. Then the disturbing function (divided by the mass of the disturbing planet) is

$$R = \{r^2 + r'^2 - 2rr' \cos S\}^{-\frac{1}{2}} - \frac{r \cos S}{r'^2}$$

where $\cos S = \cos(-f + f' - \omega + \omega') - \sigma^2\{\cos(-f + f' - \omega + \omega') - \cos(f + f + \omega + \omega')\}$

If in this expression we replace r by a, r' by a', we have

$$R = \{a^2 + a'^2 - 2aa' \cos S\}^{-\frac{1}{2}} - \frac{a \cos S}{a'^2}$$

3. Suppose we have an expression

$$\phi(a) e^{\mathrm{I} f \sqrt{-1}}$$

and we require to replace a by r and f by its value in terms of g, and we wish to have the modified function expressed in a series of multiples of $e^{g\sqrt{-1}}$, we may adopt the following notation :

$$D = a \frac{d}{da},$$

$$\phi(r) e^{\mathrm{I} f \sqrt{-1}} =$$

$$\sum \sum \frac{\epsilon^n}{\left(\frac{n+j}{2}\right)! \left(\frac{n-j}{2}\right)!} \{n, j, \mathrm{I}, D\} e^{(\mathrm{I}+j) g \sqrt{-1}}$$

where the quantity in brackets $\{n, j, \mathrm{I}, D\}$ is an operator operating